Computational Analysis of Coffee Burns: 1-D Heat Transfer in the Skin

Christopher Ahn
David Berry
Tony Yen

BENG221
10/19/12

Problem Statement

In 1992, Stella Liebeck (age 79) was in the passenger seat of her grandson's car when she tried to add milk and sugar to her recently purchased McDonalds coffee. She placed the coffee cup between her thighs and tried to pull off the lid to the coffee. She proceeded to spill the hot coffee onto her thighs where her sweatpants quickly absorbed the hot liquid and kept it on her skin. She suffered third degree burns and spent weeks in the hospital as a result of the burn. She sued McDonalds and was initially awarded $2.7 million in damages.

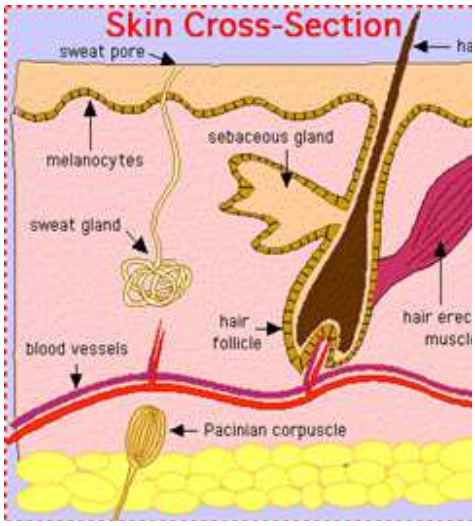

Figure 1. Cross Section of the skin (1)

The focus of this study was to analyze the 1-dimensional heat transfer of hot coffee spilled on the skin through the dermis to be used to predict third degree burns. The skin is a multilayer tissue that can be simplified into a dermal and epidermal layer (Figure 1). The different layers of the skin comprise of different molecules, cells and glands and therefore have different thermal properties that describe rate of heat transfer through the skin. The thermal properties will be integral to solving the 1-dimensional heat equation later. The equation will be analytically solved using two thermal conductivity values (k), corresponding to the epidermis and dermis respectively.

Also important when solving the 1-dimensional heat equation are the boundary conditions. At the base of the dermis lie the capillaries which can be treated as a heat sink that remains at ambient body temperature (37˚C). The temperature of the epidermis-coffee boundary relies entirely on the temperature of the coffee that is being spilled on the skin. For this study, the heat-transfer though the skin was analyzed at two relevant temperatures for serving coffee. The Journal of Food Science determined the average temperature a consumer enjoys drinking their coffee at is 60˚C (140˚F) and the optimal brewing temperature for coffee to yield the most flavor is 82˚C (180˚F). Therefore, this study will analyze the temperature transfer through the skin at both temperatures which will represent freshly brewed coffee (similar to what spilled on Stella Liebeck) and coffee that has been allowed to cool. Additionally it will be assumed that the initial temperature across the skin boundary will be equal to ambient body temperature.

In order to analytically and numerically solve the 1-dimensional heat equation, several simplifications were made:

1. The 1-dimensional heat equation can be used because we are only interested in the depth of heat penetration since this is the best way to approximate the depth of the burn. The literature describes the relationship between temperature, time and degree of burn and will be compared to the results of the study.

2. The thermal conductivity value will be assigned assuming the epidermis is the predominant factor affecting heat transfer (k=0.209 W/mK) or the dermis is the predominant factor (k=0.322W/mK) (3).
3. The temperature at the skin-capillary boundary will be 60˚C (ambient body temperature) for all time. This means the capillary is able to remove heat from the skin at the same rate the heat is applied at the skin-capillary boundary.
4. The temperature at the skin-coffee boundary will stay constant over the first four seconds. This is an appropriate approximation because the coffee was spilled into cotton sweatpants, which will absorb the coffee and keep it close to the skin. After four seconds the coffee may undergo a significant temperature drop and the person will have had time to react and brush the coffee off of the sweatpants.
5. The epidermis + dermis layer of the skin is 2 mm thick. In reality, the average thickness of skin for a Caucasian woman is 1.71mm (4). This approximation allows for ease of analysis.

Analytical Solution

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{dx^2}$$

$$Boundary\ Conditions: u(0,t) = T_1 \quad u(L,t) = T_2 \quad \rightarrow for\ t > 0$$

$$Initial\ Condition: u(x,0) = T_1$$

$$Separation\ of\ variables\ will\ not\ work\ because\ boundary\ conditions\ are\ not\ homogeneous.$$

$$Eventually\ the\ temperature\ distribution\ will\ stabilize\ and\ we\ have\ steady\ state$$

$$\lim_{t\to\infty} u(x,t) = u_{inf}(x)$$

$$Where\ u_{inf}(x)\ is\ the\ equilibrium\ temperature.\ Therefore\ this\ should\ satisfy.$$

$$0 = \frac{\partial^2 u_{inf}}{\partial x^2} \qquad u_{inf}(0) = T_1 \qquad u_{inf}(L) = T_2$$

$$u_{inf}(x) = Ax + B$$

$$u_{inf}(0) = T_1 = 0 + B, \quad B = T_1$$

$$u_{inf}(L) = T_2 = A * L + T_1, \quad A = \frac{T_2 - T_1}{L}$$

$$u_{inf}(x) = \frac{T_2 - T_1}{L}\ x + T_1 \ \leftarrow steady\ state\ equation$$

$$Lets\ define\ the\ function: v(x,t) = u(x,t) - u_{inf}(x)$$

$$u(x,t) = v(x,t) + u_{inf}(x)$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial u_{inf}}{\partial t} = \frac{\partial v}{\partial t} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u_{inf}}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

$$I.C. = v(x,0) = u(x,0) - u_{inf}(x) = T_1 - \left(\frac{T_2 - T_1}{L}\ x + T_1\right) = \frac{T_1 - T_2}{L}\ x$$

$$B.C. = v(0,t) = u(0,t) - u_{inf}(0) = T_1 - T_1 = 0$$

$$B.C. = v(L,t) = u(L,t) - u_{inf}(L) = T_2 - T_2 = 0$$

$$Now\ we\ have\ homogeneous\ boundary\ conditions!!!!$$

$$Separate\ equations: v(x,t) = \varphi(x)G(t)$$

$$\varphi(x)\frac{\partial G(t)}{\partial t} = k\ \frac{\partial^2 \varphi(x)}{\partial x^2}$$

$$\frac{\frac{\partial^2 \varphi(x)}{\partial x^2}}{\varphi(x)} = \frac{1}{k}\frac{\frac{\partial G(t)}{\partial t}}{G(t)} = -\lambda$$

$$\frac{\partial G(t)}{\partial t} = -k\lambda G(t)$$

$$G(t) = c * e^{-k\lambda t}$$

$$\frac{\partial^2 \varphi(x)}{\partial x^2} + \lambda\varphi(x) = 0$$

$$\lambda > 0$$

$$\varphi(x) = A * \cos\left(\sqrt{\lambda}x\right) + B * \sin\left(\sqrt{\lambda}x\right)$$

$$Solve\ using\ Boundary\ Conditions: \varphi(0) = 0 = A$$

$$\varphi(L) = 0 = B * \sin\left(\sqrt{\lambda}L\right)$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi(x)_n = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1,2,3,\ldots$$

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L \left(\frac{T_1 - T_2}{L}\ x\right)\sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1,2,3,\ldots$$

$$B_n = \frac{2(T_1 - T_2)(-1)^n}{\pi n} \qquad n = 1,2,3,\ldots$$

$$Plugging\ back\ into\ the\ original\ equation:$$

$$u(x,t) = u_{inf}(x) + v(x,t)$$

$$u(x,t) = \frac{T_2 - T_1}{L}\, x + T_1 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Numerical Solution

For the numerical solution, a finite difference method was used, which was the average of the forward difference and backwards difference models. The following figures show the results of our analytical solutions and numerical solutions.

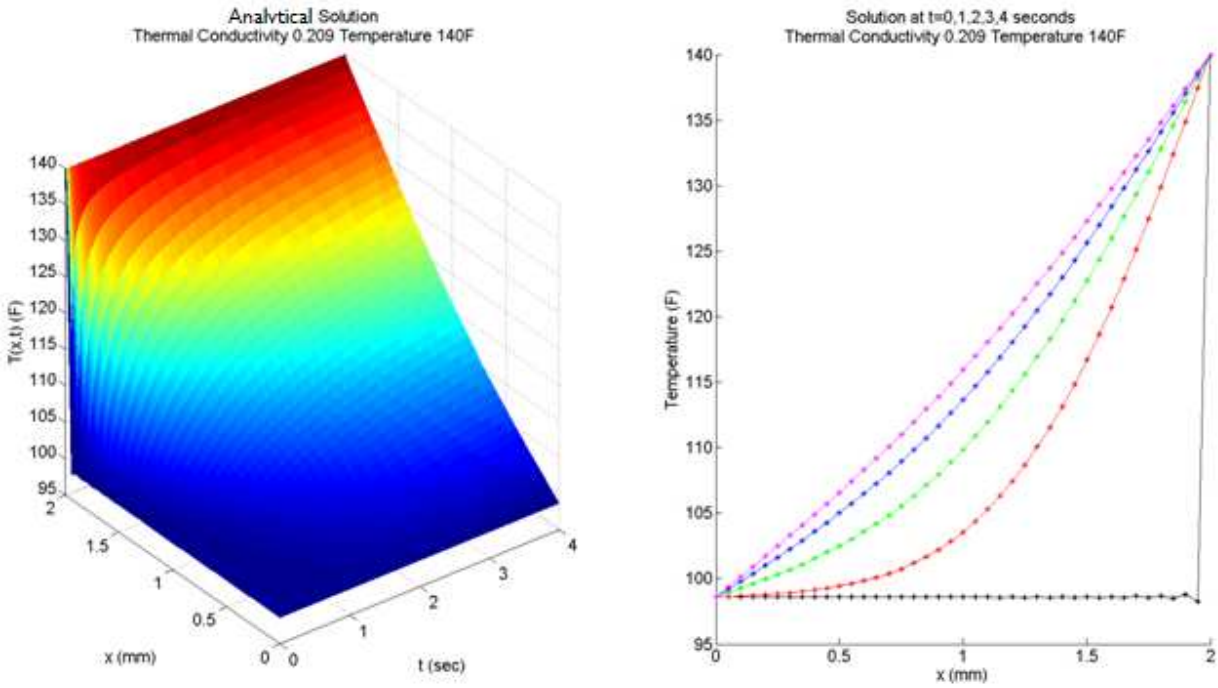

Figure 2. Analytical Solution for a Thermal Conductivity of 0.209 W/mK (epidermis dominated) and coffee temperature 60˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

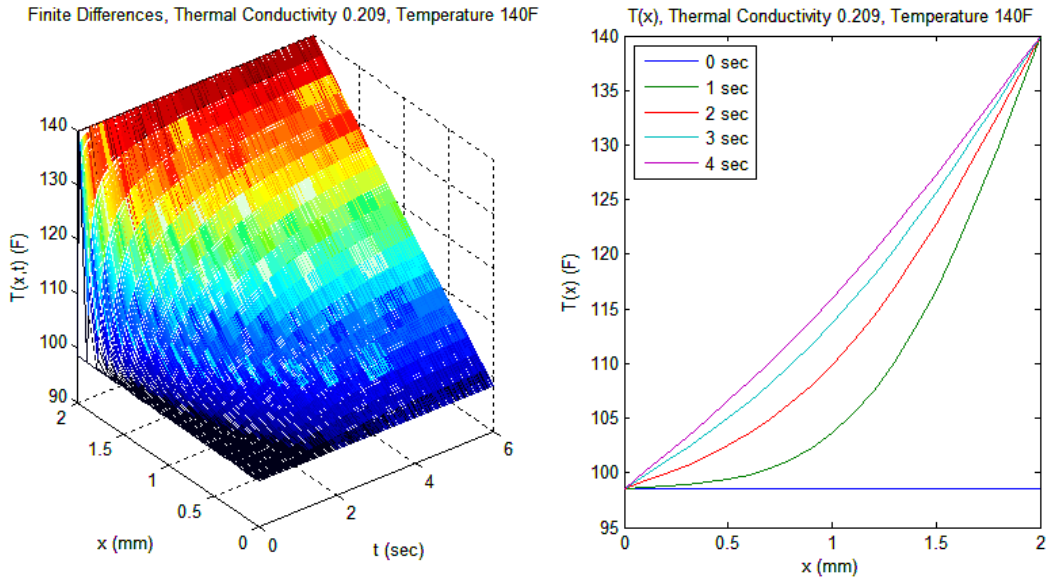

Figure 3. Finite Difference solution for a Thermal Conductivity of 0.209 W/mK (epidermis dominated) and coffee temperature 60˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

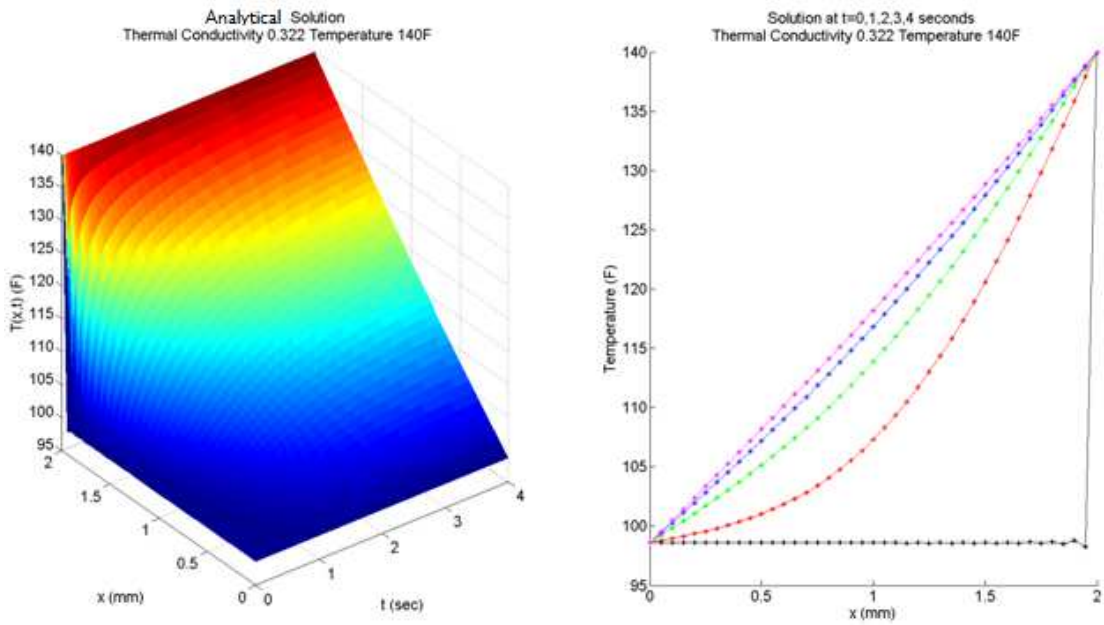

Figure 4. Analytical Solution for a Thermal Conductivity of 0.322 W/mK (dermis dominated) and coffee temperature 60˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

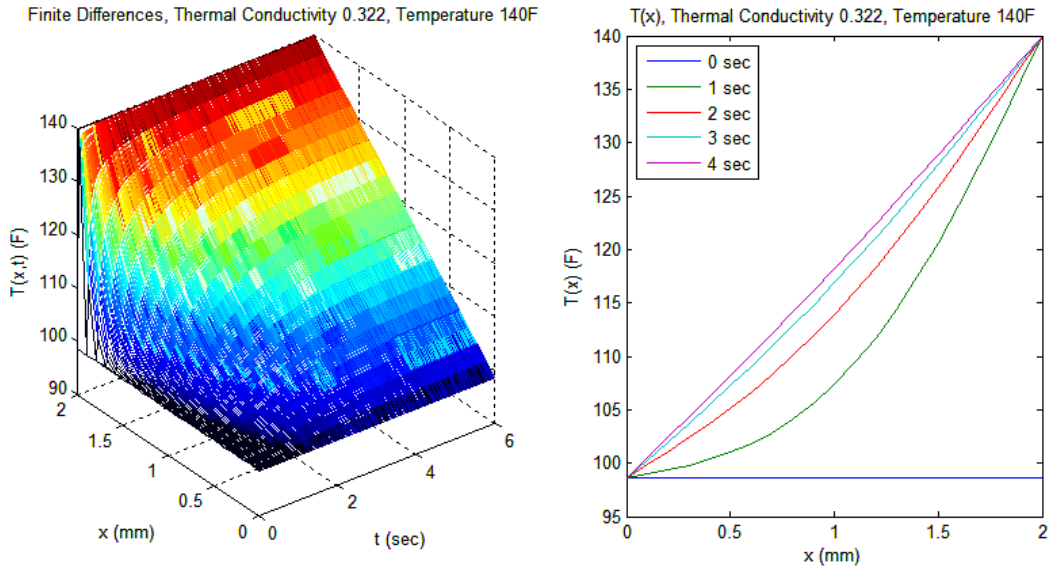

Figure 5. Finite Difference Solution for a Thermal Conductivity of 0.322 W/mK (dermis dominated) and coffee temperature 60˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

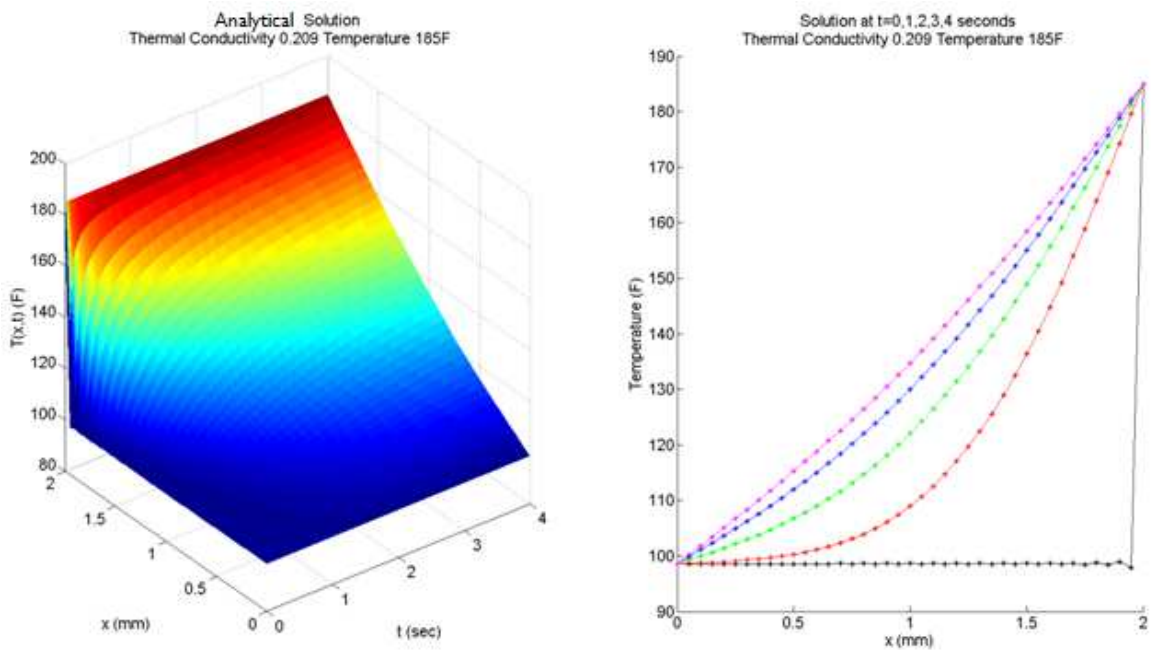

Figure 6. Analytical Solution for a Thermal Conductivity of 0.209 W/mK (epidermis dominated) and coffee temperature 82˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

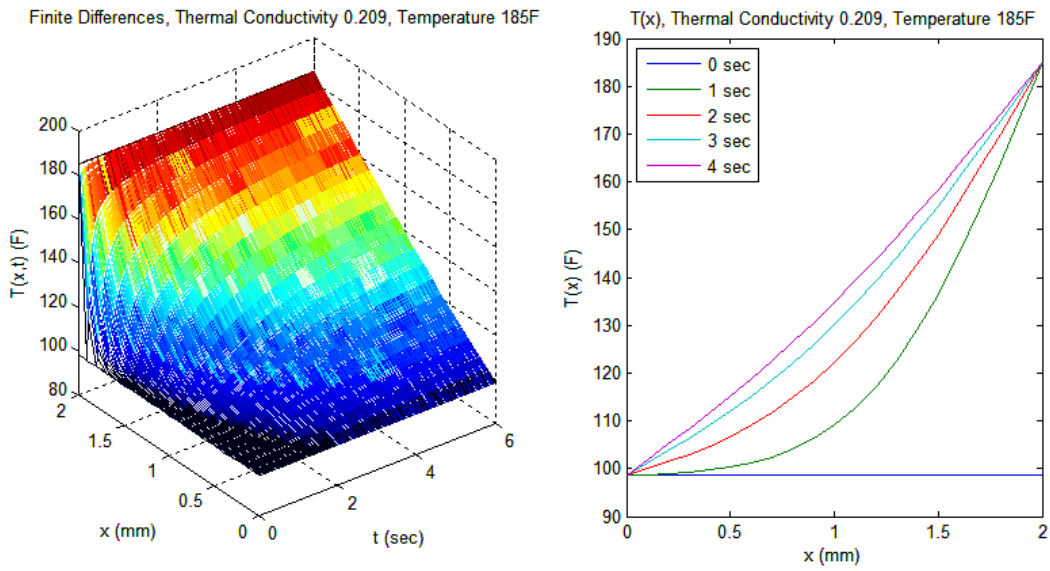

Figure 7. Finite Difference Solution for a Thermal Conductivity of 0.209 W/mK (epidermis dominated) and coffee temperature 82˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

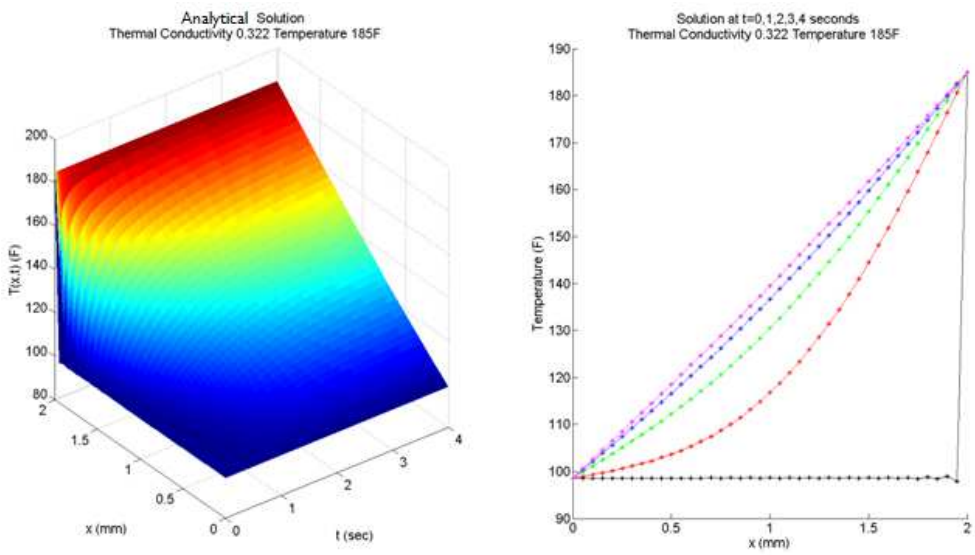

Figure 8. Analytical Solution for a Thermal Conductivity of 0.322 W/mK (dermis dominated) and coffee temperature 82˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

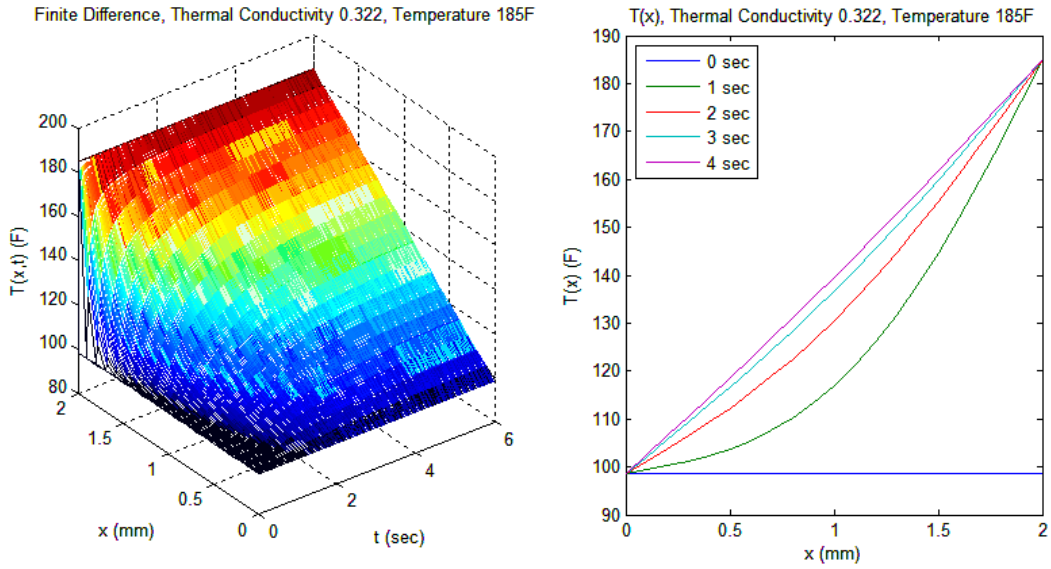

Figure 9. Finite Difference Solution for a Thermal Conductivity of 0.322 W/mK (epidermis dominated) and coffee temperature 82˚C. The left graph shows the temperature profile over the first four seconds, the right graphs shows the temperature profiles at t= 0s, 1s, 2s, 3s, 4s

Limitations

This study, while providing a powerful estimate of burn depth, is limited by the assumptions that were established to solve the partial differential equation. The most powerful assumption was the boundary condition at the capillary-skin boundary. We stated the boundary was to remain constant at ambient body temperature. In reality, the heat can penetrate down to and past the depth of the capillary layer. Additionally, blood flow in the capillaries will not be able to remove heat at the same rate at which it is added to the system at that location in space. Additionally, a constant boundary condition for the skin-coffee layer introduces error into this simulation. The coffee will begin to cool down the moment it is absorbed by the sweatpants. We could have estimated a time-dependent rate of heat loss for the coffee absorbed in the sweatpants and applied it to our system. However, that complicated the boundary conditions and our assumption the coffee temperature stayed constant seemed appropriate.

Our last major assumption was the thermal conductivity of the system remained constant through both layers. We were able to make this assumption by stating the heat transfer through the skin was largely dependent on either the epidermis or the dermis not a combination of both. One solution to bypass the error associated with this assumption is to determine a series of two linked partial differential equations both with one constant boundary condition and one flux boundary condition, and assign a thermal conductivity value to each equation, representing the epidermis and dermis respectively. This way the dermis temperature profile is directly related to

the temperature profile of the epidermis. However, this creates a series of nasty, time dependent, mixed boundary condition equations which is much more difficult to solve.

Conclusions

In this study we have established a proof-of-concept for the one-dimensional modeling of heat transfer through skin in order to determine the depth of severe skin burns. A literature search shows that severe burns will occur at approximately one half-second after the skin reaches a temperature of 160 ºF, or 71.1 ºC. According to the predictions of our model, 60 ºC coffee (comfortable drinking temperature) would not create severe burns during the time we inspected, however, 82 ºC coffee (freshly brewed) would create severe burns at either 3.5 or 4.5 seconds, depending on the heat transfer coefficient selected , with the lower heat transfer coefficient of 0.209 W/mK resulting in 4.5 seconds before a burn, and the higher coefficient of 0.322 W/mK resulting in 3.5 seconds before a burn. This would indicate that coffee brewed according to hospitality standards – that is, 82 ºC – has the potential to cause severe burning, but coffee at the temperature that most people find comfortable to drink a hot beverage, around 60 ºC, would not cause a burn.

Our finite difference method did result in a good approximation of our analytical solution, likely because our model was fairly simple. Should additional complexity be introduced – for example, by using a two-equation PDE system reflecting both the dermis and the epidermis – the finite difference method may not be as accurate, and a method such as MATLAB's pdepe may be required.

While additional parameters and equations would be required to establish a more realistic model, we find that our simple model does give important information regarding a time range for severe burning, and serves as a proof of concept should we wish to explore further modeling (and eventually compare our model results to some sort of real-world experimental testing).

References

1. "Skin Anatomy." *EnchantedLearning.com*. Web. 7 Oct. 2012. <http://www.enchantedlearning.com/subjects/anatomy/skin/>.
2. H.S. Lee, M. O'Mahony. At what temperatures do consumers like to drink coffee?: mixing methods. J Food Sci, 67 (2002), pp. 2774–2777
3. Lipkin, Martin, and James D. Hardy. "Measurement of Some Thermal Properties of Human Tissues." *Journal of Applied Physiology* 7.10 (1954): 212-17.
4. Alexander, Harold, and D. L. Miller. "Determining Skin Thickness with Pulsed Ultrasound."*Journal of Investigative Dermatology* 71.1 (1979): 17-19.
5. Accurate Building Inspectors, Inc. "Time/Temperature to produce 2nd and 3rd Degree Burns." December 15, 2005. Accessed 10/5/2012. http://www.accuratebuilding.com/services/legal/charts/hot_water_burn_scalding_graph.html

```
% BENG 221 Project - Analytical Solution
% Chris Ahn, David Berry, Tony Yen
% 10/10/2012
tic
clear all, close all, clc
k = [0.209 0.322]; %(W/mK) for epidermis
L = 2; %mm
T_2 = 273.15+([185 140]-32)*5/9;  %kelvin
T_1 = 273.15+(98.6-32)*5/9; %kelvin

t_step = 0.01; %sec
t_tot =4;
x_step = 0.05; %mm

B_n = @(T_1, T_2, n) 2*(T_2-T_1)*((-1)^n)/(pi*n);
B_0 = @(T_1, T_2, x) (T_2-T_1)*x/L+T_1;
u = @(n,x,t,k,T_1,T_2) B_n(T_1,T_2,n).*sin(n*pi*x/L).*exp(-k*t*(n*pi/(L))^2);
for m=1:length(T_2)
    for j=1:length(k)
        [x,t] = meshgrid(0:x_step:L,0:t_step:t_tot);
        U = zeros(size(B_0(T_2(m),T_1,x)));
        for i = 1:1000
            U = U + u(i,x,t,k(j),T_1,T_2(m));
        end
        figure
        subplot(1,2,1)
        T_sol = B_0(T_1,T_2(m),x) + U;
        T_sol=((T_sol-273.15)*9/5+32)';
        mesh(t,x,T_sol);
        title(sprintf('Thermal Conductivity %1.3f \nTemperature %3.2fK',k(j),T_2(m)))
        ylabel('x (mm)')
        xlabel('t (sec)')
        zlabel('T(x,t) (K)')
        subplot(1,2,2), hold on
        plot(x,T_sol(1,:),'k*-',x,T_sol(100,:),'r*-',x,T_sol(200,:),...
            'g*-',x,T_sol(300,:),'b*-',x,T_sol(400,:),'m*-')
        legend('t=0s','t=1s','t=2s','t=3s','t=4s','Location','Best')
        xlabel('x (mm)')
        ylabel('Temperature (K)')
        title(sprintf('Thermal Conductivity %1.3f \nTemperature %3.2fK',k(j),T_2(m)))
    end
end
toc
```

```
% BENG 221 Group Project
% Christopher Ahn, David Berry and Tony Yen
% Numerical Approximation Code

clear all;
close all;

%% k = 0.209, T = 140 F
k = 0.209; %(W/mmk) for epidermis
L = 2; %mm
T_2 = 273.15+(140-32)*5/9;  %kelvin
T_1 = 273.15+(98.6-32)*5/9; %kelvin

% x&t boundry
x_step = 0.1;
t_step = 0.01; %sec
t_tot =6;

xmesh = 0:x_step:L;
tmesh = 0:t_step:t_tot;

% Stepsizes
n_x = length (xmesh);
n_t = length (tmesh);
stepsize = k*t_step / x_step^2;
T = zeros(n_x,n_t);

T(1,:)= T_1;% Boundry Condition
T(n_x,:)=T_2;
T(:,1)=T_1; % Initial Condition

for t = 1:n_t-1
    for x = 2:n_x-1  %everything in the middle
        T(x,t+1) = T(x,t) + stepsize*(T(x+1,t) - 2*T(x,t) + T(x-1,t));
    end
end

T=(T-273.15)*9/5+32; %back from K to F

figure
subplot(1,2,1)
mesh(tmesh,xmesh,T);
title('Finite Differences, Thermal Conductivity 0.209, Temperature 140F');
xlabel('t (sec)');
ylabel('x (mm)');
zlabel('T(x,t) (F)');

subplot(1,2,2)
plot(xmesh, T(:,0/t_step+1),xmesh,T(:,1/t_step+1),xmesh,T(:,2/t_step+1),xmesh,T(:,↙
3/t_step+1),xmesh,T(:,4/t_step+1))
title('T(x), Thermal Conductivity 0.209, Temperature 140F');
legend('0 sec','1 sec','2 sec','3 sec','4 sec', 'Location', 'NorthWest');
xlabel('x (mm)');
ylabel('T(x) (F)');
```

```
%% k = 0.322, T = 140 F

k = 0.322; %(W/mmk) for epidermis
L = 2; %mm
T_2 = 273.15+(140-32)*5/9;  %kelvin
T_1 = 273.15+(98.6-32)*5/9; %kelvin

% x&t boundry
x_step = 0.1;
t_step = 0.01; %sec
t_tot =6;

xmesh = 0:x_step:L;
tmesh = 0:t_step:t_tot;

% Stepsizes
n_x = length (xmesh);
n_t = length (tmesh);
stepsize = k*t_step / x_step^2;
T = zeros(n_x,n_t);

T(1,:)= T_1;% Boundry Condition
T(n_x,:)=T_2;
T(:,1)=T_1; % Initial Condition

for t = 1:n_t-1
    for x = 2:n_x-1  %everything in the middle
        T(x,t+1) = T(x,t) + stepsize*(T(x+1,t) - 2*T(x,t) + T(x-1,t));
    end
end

T=(T-273.15)*9/5+32; %back from K to F

figure
subplot(1,2,1)
mesh(tmesh,xmesh,T);
title('Finite Differences, Thermal Conductivity 0.322, Temperature 140F');
xlabel('t (sec)');
ylabel('x (mm)');
zlabel('T(x,t) (F)');

subplot(1,2,2)
plot(xmesh, T(:,0/t_step+1),xmesh,T(:,1/t_step+1),xmesh,T(:,2/t_step+1),xmesh,T(:,↙
3/t_step+1),xmesh,T(:,4/t_step+1))
title('T(x), Thermal Conductivity 0.322, Temperature 140F');
legend('0 sec','1 sec','2 sec','3 sec','4 sec', 'Location', 'NorthWest');
xlabel('x (mm)');
ylabel('T(x) (F)');

%% k = 0.209, T = 185 F
k = 0.209; %(W/mmk) for epidermis
L = 2; %mm
T_2 = 273.15+(185-32)*5/9;  %kelvin
T_1 = 273.15+(98.6-32)*5/9; %kelvin
```

```
% x&t boundry
x_step = 0.1;
t_step = 0.01; %sec
t_tot =6;

xmesh = 0:x_step:L;
tmesh = 0:t_step:t_tot;

% Stepsizes
n_x = length (xmesh);
n_t = length (tmesh);
stepsize = k*t_step / x_step^2;
T = zeros(n_x,n_t);

T(1,:)= T_1;% Boundry Condition
T(n_x,:)=T_2;
T(:,1)=T_1; % Initial Condition

for t = 1:n_t-1
    for x = 2:n_x-1  %everything in the middle
        T(x,t+1) = T(x,t) + stepsize*(T(x+1,t) - 2*T(x,t) + T(x-1,t));
    end
end

T=(T-273.15)*9/5+32; %back from K to F

figure
subplot(1,2,1)
mesh(tmesh,xmesh,T);
title('Finite Differences, Thermal Conductivity 0.209, Temperature 185F');
xlabel('t (sec)');
ylabel('x (mm)');
zlabel('T(x,t) (F)');

subplot(1,2,2)
plot(xmesh, T(:,0/t_step+1),xmesh,T(:,1/t_step+1),xmesh,T(:,2/t_step+1),xmesh,T(:,↙
3/t_step+1),xmesh,T(:,4/t_step+1))
title('T(x), Thermal Conductivity 0.209, Temperature 185F');
legend('0 sec','1 sec','2 sec','3 sec','4 sec', 'Location', 'NorthWest');
xlabel('x (mm)');
ylabel('T(x) (F)');

%% k = 0.322, T = 185 K
k = 0.322; %(W/mmk) for epidermis
L = 2; %mm
T_2 = 273.15+(185-32)*5/9;  %kelvin
T_1 = 273.15+(98.6-32)*5/9; %kelvin

% x&t boundry
x_step = 0.1;
t_step = 0.01; %sec
t_tot =6;

xmesh = 0:x_step:L;
tmesh = 0:t_step:t_tot;
```

```
% Stepsizes
n_x = length (xmesh);
n_t = length (tmesh);
stepsize = k*t_step / x_step^2;
T = zeros(n_x,n_t);

T(1,:)= T_1;% Boundry Condition
T(n_x,:)=T_2;
T(:,1)=T_1; % Initial Condition

for t = 1:n_t-1
    for x = 2:n_x-1  %everything in the middle
        T(x,t+1) = T(x,t) + stepsize*(T(x+1,t) - 2*T(x,t) + T(x-1,t));
    end
end

T=(T-273.15)*9/5+32; %back from K to F

figure
subplot(1,2,1)
mesh(tmesh,xmesh,T);
title('Finite Difference, Thermal Conductivity 0.322, Temperature 185F');
xlabel('t (sec)');
ylabel('x (mm)');
zlabel('T(x,t) (F)');

subplot(1,2,2)
plot(xmesh, T(:,0/t_step+1),xmesh,T(:,1/t_step+1),xmesh,T(:,2/t_step+1),xmesh,T(:,↙
3/t_step+1),xmesh,T(:,4/t_step+1))
title('T(x), Thermal Conductivity 0.322, Temperature 185F');
legend('0 sec','1 sec','2 sec','3 sec','4 sec', 'Location', 'NorthWest');
xlabel('x (mm)');
ylabel('T(x) (F)');
```